Fibonacci Book
Rabbit Problem

Some problems to solve

Solutions

Binet's Formula

Fibonacci Numbers

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Bard Math Circle AMC 8
November 12, 2019

Fibonacci's Rabbit Problem

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

—A problem from the third section of Liber abaci (1202).

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Recursive Definition

$$\mathrm{F(Next)} = \mathrm{F(Current)} + \mathrm{F(Productive)}, \qquad \mathrm{F}(0) = \mathrm{F}(1) = 1$$

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Rabbit Population

Month:	0	1	2	3	4	5	6	7	8	9	10
Rabbits:	1	1									

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Month:	0	1	2	3	4	5	6	7	8	9	10
Rabbits:	1	1	2								

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Month:	0	1	2	3	4	5	6	7	8	9	10
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Month:	0	1	2	3	4	5	6	7	8	9	10
Rabbits:	1	1	2	3	5	8	13	21	34	55	89

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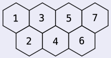
Solve one of these problems:

A **composition** of n is a way to write n as the sum of positive integers (order matters). How many compositions are there of 8 that don't use 1?

How many *increasing* paths are there through the honeycomb from 1 to 7?



How many compositions are there of 7 into odd parts?

How many ways are there to climb a set of 6 stairs, one or two steps at a time?

How many subsets are there of $\{1, 2, 3, 4, 5\}$ that include no two consecutive numbers?

How many binary sequences of length 5 are there, with no consecutive 0's?

In how many ways can you tile a 2×6 rectangle with 2×1 dominoes?

Find 6 positive integer solutions (x, y) of $y^2 - xy - x^2 = \pm 1$.

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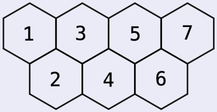

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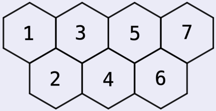
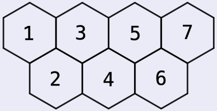


Path(7)

How many *increasing* paths are there through the honeycomb from 1 to 7?



Path(6) Path(7)

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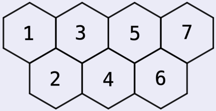

Path(5)   Path(6)   Path(7)

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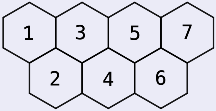

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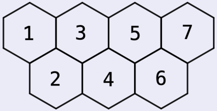

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Path(n)

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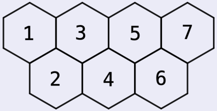

$\cdots$ Path(5) Path(6) Path(7)

Path(n)

n	1	2	3	4	5	6	7
$P(n)$							

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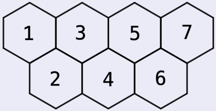

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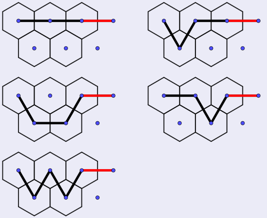
Paths ending 5-7



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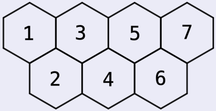

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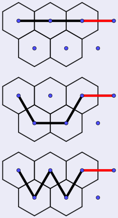
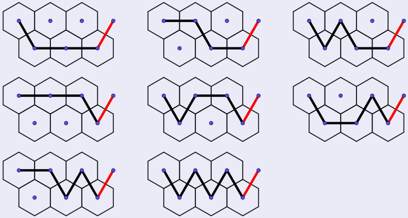


Paths ending 6-7



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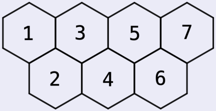
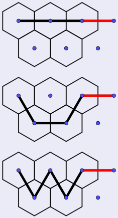


Path(5)+Path(6)=Path(7)

Path(n)

n	1	2	3	4	5	6	7
$P(n)$							

Paths ending 5-7



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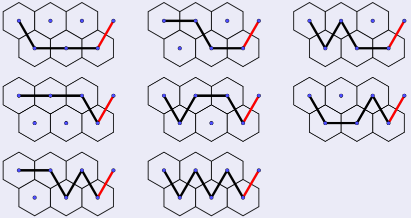

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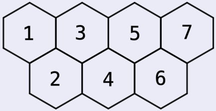

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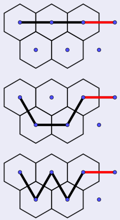

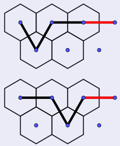

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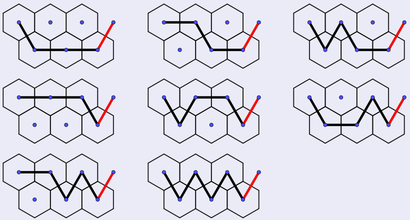

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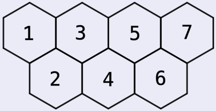
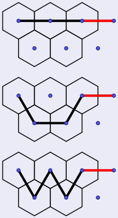


$$\text{Path}(5) + \text{Path}(6) = \text{Path}(7)$$

$$\text{Path}(n-2) + \text{Path}(n-1) = \text{Path}(n)$$

n	1	2	3	4	5	6	7
$P(n)$	1	1					

Paths ending 5-7



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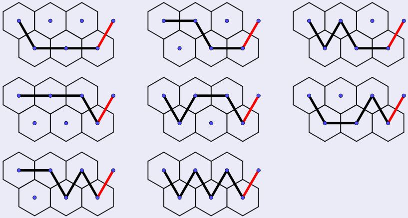

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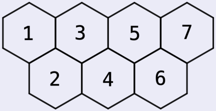

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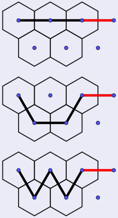

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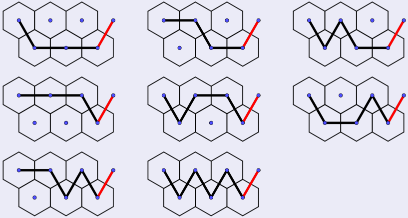

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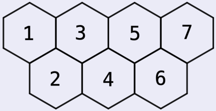

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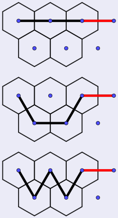

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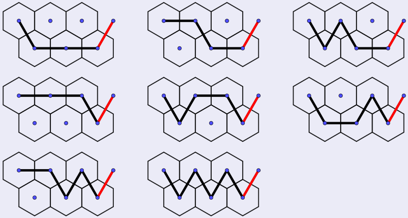

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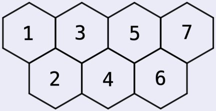

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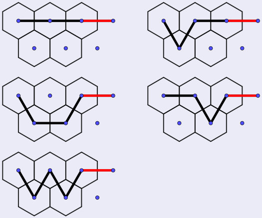

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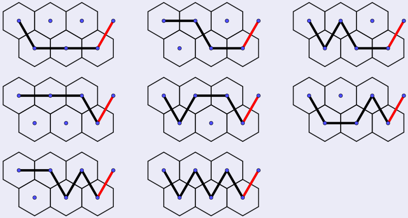

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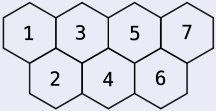

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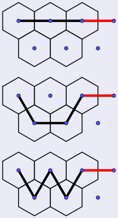

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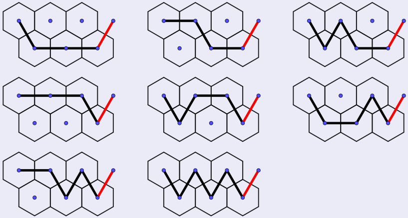

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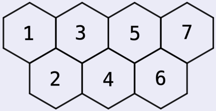

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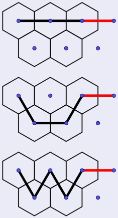

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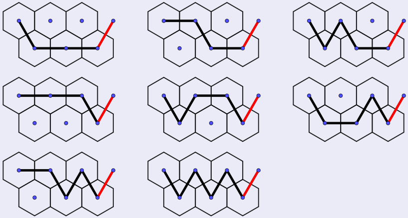

The Fibonacci Sequence

$F(n) = F(n-1) + F(n-2)$ (for $n > 2$), $F(1) = 1$, $F(2) = 1$

n	0	1	2	3	4	5	6	$\cdots$
$F(n)$	0	1	1	2	3	5	8	$\cdots$

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A Fibonacci-ish Sequence (Gibonacci?)

$G(n) = G(n-1) + G(n-2)$

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	4	-2	2	0	2	2	4	$\cdots$

Fact 1: Scaling a Fibonacci-ish Sequence yields a ⋯

n	0	1	2	3	4	5	6	⋯
$G(n)$	4	-3	1	-2	-1	-3	-4	⋯
$4G(n)$	16	-12	4	-8	-4	-12	-16	⋯

Fact 1: Scaling a Fibonacci-ish Sequence yields a $\cdots$

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	4	-3	1	-2	-1	-3	-4	$\cdots$
$4G(n)$	16	-12	4	-8	-4	-12	-16	$\cdots$

Fact 2: If $G(0)$ is 0 then ...

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	0	3	3	6	9	15	24	$\cdots$

Fact 1: Scaling a Fibonacci-ish Sequence yields a $\cdots$

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	4	-3	1	-2	-1	-3	-4	$\cdots$
$4G(n)$	16	-12	4	-8	-4	-12	-16	$\cdots$

Fact 2: If $G(0)$ is 0 then ...

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	0	3	3	6	9	15	24	$\cdots$
$F(n)$	0	1	1	2	3	5	8	$\cdots$

Fact 1: Scaling a Fibonacci-ish Sequence yields a $\cdots$

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	4	-3	1	-2	-1	-3	-4	$\cdots$
$4G(n)$	16	-12	4	-8	-4	-12	-16	$\cdots$

Fact 2: If $G(0)$ is 0 then ...

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	0	3	3	6	9	15	24	$\cdots$
$F(n)$	0	1	1	2	3	5	8	$\cdots$

Fact 3: Subtracting Fibonacci-ish Sequences yields a ...

n	0	1	2	3	4	5	$\cdots$
$G(n)$	4	3	7	10	17	27	$\cdots$
$H(n)$	2	1	3	4	7	11	$\cdots$

Fact 1: Scaling a Fibonacci-ish Sequence yields a $\cdots$

n	0	1	2	3	4	5	6	$\cdots$
$G(n)$	4	-3	1	-2	-1	-3	-4	$\cdots$
$4G(n)$	16	-12	4	-8	-4	-12	-16	$\cdots$

Fact 2: If $G(0)$ is 0 then ...

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$G(n)$	4	3	7	10	17	27	$\cdots$
$H(n)$	2	1	3	4	7	11	$\cdots$
$G(n) - H(n)$	2	2	4	6	10	16	$\cdots$

Two interesting Fibonacci-ish sequences

n	0	1	2	3	4	$\cdots$
$R(n)$	1	r	r^2	r^3	r^4	$\cdots$
$S(n)$	1	s	s^2	s^3	s^4	$\cdots$

Two interesting Fibonacci-ish sequences

n	0	1	2	3	4	$\cdots$
$R(n)$	1	r	r^2	r^3	r^4	$\cdots$
$S(n)$	1	s	s^2	s^3	s^4	$\cdots$

Necessary (and Sufficient) Conditions

Two interesting Fibonacci-ish sequences

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Necessary (and Sufficient) Conditions

$R(0) + R(1) = R(2)$ or $1 + r = r^2$ and also $1 + s = s^2$

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$r = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $s = \frac{1-\sqrt{5}}{2} \approx -0.618$

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$R(n)$	1	r	r^2	r^3	r^4	$\cdots$
$S(n)$	1	s	s^2	s^3	s^4	$\cdots$
$R(n)-S(n)$	0	$r-s$	r^2-s^2	r^3-s^3	r^4-s^4	$\cdots$

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$R(n)$	1	r	r^2	r^3	r^4	$\cdots$
$S(n)$	1	s	s^2	s^3	s^4	$\cdots$
$R(n) - S(n)$	0	$r - s$	$r^2 - s^2$	$r^3 - s^3$	$r^4 - s^4$	$\cdots$
$F(n)$	0	1	$\frac{r^2-s^2}{r-s}$	$\frac{r^3-s^3}{r-s}$	$\frac{r^4-s^4}{r-s}$	$\cdots$

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$F(n)$	0	1	$\frac{r^2-s^2}{r-s}$	$\frac{r^3-s^3}{r-s}$	$\frac{r^4-s^4}{r-s}$	$\cdots$

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Binet's Formula

$$F(n)=\frac{r^n-s^n}{r-s}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Fibonacci Book Rabbit Problem
Some problems to solve
Solutions
Binet's Formula

The End

Thank You!

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